Malaysian Journal of Mathematical Sciences 15(3): 357–368 (2021)

Malaysian Journal of Mathematical Sciences

Journal homepage: https://einspem.upm.edu.my/journal

On LA-Semimodule Over LA-Semiring

Andari, A. * and Rouf, A.

Department of Mathematics, Brawijaya University, Indonesia

E-mail: ari_mat@ub.ac.id
**Corresponding author*

Received: 30 October 2019
Accepted: 20 May 2021

Abstract

In this paper, we develop an LA-module over LA-ring to a new concept namely LA-semimodule over LA-semiring. Let S be a non-empty set with two binary operations "$+$" and "$*$". Set S is called a left almost semiring (LA-semiring) if $(S,+)$ is an LA-semigroup, $(S,*)$ is an LA-semigroup and satisfying left and right distributive law of "$*$" over "$+$" hold. Let $(S,+,*)$ is an LA-semiring with left additive identity equal to 0_S and left multiplicative identity equal to 1, non-empty set M is called an LA-semimodule over S if 1) $(M,+)$ is an LA-semigroup with left identity, 2) the map $S \times M \rightarrow M$, $(s,m) \mapsto sm$ where $s \in S$ and $m \in M$ satisfies i) $s(m+n) = sm+sn$, ii) $(r+s)m = rm+sm$, iii) $r(sm) = s(rm)$, iv) $1*m = m$, for all $r,s \in R$, and $m,n \in M$. Then, we investigate the basic properties and the Isomorphims Theorem for LA-semimodule over LA-semiring.

Keywords: LA-semigroup; LA-semiring; LA-semimodule.

1 Introduction

The concept of AG-groupoid is a generalization of commutative semigroup concept without associative law that introduced by [4]. A grupoid S is called AG-groupoid if its element satisfy the left invertive law i.e $(ab)c = (cb)a$ for all $a, b, c \in S$. In [1], AG-groupoid is also known as left almost semigroup (LA-semigroup). A groupoid G is called medial if G satisfy the medial law i.e $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in G$. A groupoid G is called paramedial if satisfy the paramedial law i.e $(ab)(cd) = (db)(ca)$ for all $a, b, c, d \in G$ [1]. An LA-semigroup S always satisfies the medial law [[1], Lemma 1.1(i)] while an LA-semigroup S with left identity e always satisfies the paramedial law [[1], Lemma 1.2 (ii)]. An LA-semigroup S with left identity e also satisfies $a(bc) = b(ac)$ for all $a, b, c \in S$ [[6], Lemma 4].

The works of [3] and [5] extend the notion of LA-semigroup into LA-group. An LA-semigroup G is called an LA-group if there exists left identity $e \in G$ such that $ea = a$ for all $a \in G$ and for all $a \in G$ there exists $a^{-1} \in G$ such that $a^{-1}a = aa^{-1} = e$. Then, [8] give the properties of cancellative LA-semigroup. An element a of an LA-semigroup S is called left cancellative if $ax = ay$ implies $x = y$ for all $x, y \in S$. Similarly, an element a of an LA-semigroup S is called right cancellative if $xa = ya$ imples $x = y$ for all $x, y \in S$. An element a of an LA-semigroup S is called cancellative if it is both left and right cancellative. An LA-semigroup S is called left cancellative if every element of S is left cancellative. Similarly, an LA-semigroup S is called right cancellative if every element of S is right cancellative and it is called cancellative if every element of S is cancellative. A finite cancellative LA-semigroup is an LA-group [8].

In 2011, [9] extended LA-group to a non-associative structure with respect to both binary operations $'+'$ and $'\cdot'$ namely left almost ring (LA-ring). A left almost ring means a nonempty set R with at least two element such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup and both left and right distributive laws hold.

Next, [10] extended LA-group and LA-ring concept to LA-module. Let $(R, +, \cdot)$ be an LA-ring with left identity 1. An LA-group $(M, +)$ is called an LA-module over R, if the map $R \times M \to M$ is defined $(r, m) \mapsto rm \in M$, and where $r \in R$ and $m \in M$ satisfies : $r(m_1 + m_2) = rm_1 + rm_2$, $(r_1 + r_2)m = r_1m + r_2m$, $r_1(r_2m) = r_2(r_1m)$, $1 \cdot m = m$, for all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$.

Semiring S is a non-empty set with two binary operation that satisfy $(S, +)$ is monoid commutative, $(S, \cdot)$ is semigroup, both left and right distributive laws hold [2]. Then, [2] give the definition of semimodule over semiring and some property of it. After that, [7] extend LA-ring and semiring into LA-semiring. In this paper, we will generalize LA-module over LA-ring into LA-semimodule over LA-semiring. Then, we investigate the properties of LA-semimodule over LA-semiring, along with all that associated with LA-semimodule.

2 Result and Discussion

2.1 LA-Semimodule

In this section, we give the definition of an LA-semimodule over an LA-semiring. we study some examples of LA-semimodule and discuss the elementary properties of an LA-semimodule.

Definition 2.1 ([7]). *A left almost semiring (LA-semiring) is a non empty set S with two binary operations " $+$ " and " $*$ " that satisfying the following conditions:*

i. $(S, +)$ *is an LA-semigroup.*

ii. $(S, *)$ *is an LA-semigroup.*

iii. Both left and right distributive laws holds:
$x * (y + z) = x * y + x * z$
$(y + z) * x = y * x + z * x$
for all $x, y, z \in S$.

In this paper, all LA-semiring S are LA-semiring with left additive identity equal to 0_S and left multiplicative identity equal to 1.

Example 2.1. *Here some examples of LA-semiring:*

i. All LA-ring are LA-semiring.

ii. Let $S = \mathbb{Z}_n$, $n \in \mathbb{N}$ *and define binary operation*

$$\begin{aligned} \ominus : S \times S &\to S \\ (a, b) &\mapsto a \ominus b = b - a, \end{aligned}$$

and

$$\begin{aligned} * : S \times S &\to S \\ (a, b) &\mapsto a * b = ab. \end{aligned}$$

Note that for any $a, b, c \in S$, *we have*

$$\begin{aligned} (a \ominus b) \ominus c &= c - b + a \\ &= a - b + c \\ &= (c \ominus b) \ominus a. \end{aligned}$$

So, $(S, \ominus)$ *is an LA-semigroup. Furthermore, note that*

$$(a \ominus b) \ominus c = c - b + a \neq c - b - a = a \ominus (b \ominus c)$$

and

$$a \ominus b = b - a \neq a - b = b \ominus a.$$

Hence, $(S, \ominus)$ *is not a commutative semigroup. Since* $(S, *)$ *is a commutative monoid then* $(S, *)$ *is an LA-semigroup. Next, note that*

$$\begin{aligned} (a \ominus b)c = (b - a)c = bc - ac = ac \ominus bc \\ a(b \ominus c) = a(c - b) = ac - ab = ab \ominus ac \end{aligned}$$

for any $a, b, c \in S$. *Therefore,* $(S, \ominus, *)$ *is an LA-semiring.*

Definition 2.2. *Let $(S, +, *)$ be an LA-semiring. A set M is called LA-semimodule over LA-semiring S if satisfies:*

i. *$(M, +)$ is an LA-semigroup with left identity.*

ii. *Defined map $\cdot : S \times M \to M$ where $(r, m) \mapsto rm$, $r \in S$, $m \in M$ and satisfies:*

(a) $r(m + n) = rm + rn$

(b) $(r + s)m = rm + sm$

(c) $r(sm) = s(rm)$

(d) $1 \cdot m = m$, *for all $r, s \in S$ and $m, n \in M$.*

In this paper, all LA-semimodule M are LA-semimodule with left identity equal to 0_M.

Example 2.2. *Here some examples of LA-semimodule:*

i. *All LA-module over LA-ring R are LA-semimodule over R.*

ii. *All LA-semiring S are LA-semimodule over itself.*

Theorem 2.1. *Let $(M, +)$ be a cancellative LA-semimodule over LA-semiring $(S, +, *)$. Then, for all $s \in S$ and $a \in M$ satisfies:*

i. $s \cdot 0_M = 0_M$

ii. $0_S \cdot a = 0_M$

Proof. Let a be an arbitrary element in M and s be an arbitrary element in S, then the following conditions are hold:

i. Since M is an LA-semimodule with left identity 0_M then

$$\begin{aligned} s \cdot 0_M = s(0_M + 0_M) &\Leftrightarrow s \cdot 0_M = s \cdot 0_M + s \cdot 0_M \\ &\Leftrightarrow 0_M + s \cdot 0_M = s \cdot 0_M + s \cdot 0_M \end{aligned}$$

since M is cancellative then $0_M = s \cdot 0_M$.

ii. Since S is an LA-semiring with left additive identity 0_S then

$$\begin{aligned} 0_S \cdot a = (0_S + 0_S)a &\Leftrightarrow 0_S \cdot a = 0_S \cdot a + 0_S \cdot a \\ &\Leftrightarrow 0_M + 0_S \cdot a = 0_S \cdot a + 0_S \cdot a \end{aligned}$$

since M is cancellative then $0_M = 0_S \cdot a$.

□

2.2 LA-Subsemimodule

In this section, we give the definition of an LA-subsemimodule of LA-semimodule. Then, we initiate the following definition.

Definition 2.3. *Let M be an LA-semimodule over LA-semiring S and N be a non empty subset of M. LA-subsemigroup N is called LA-subsemimodule over S, if $SN \subseteq N$, i.e , $sn \in N$, for all $s \in S$ and $n \in N$.*

Remark 2.1. *Let M be an LA-semimodule over LA-semiring S. Then M it self and $\{0\}$ are LA-subsemimodule over LA-semiring S and its called improper LA-subsemimodule.*

Corollary 2.1. *Let M be a cancellative LA-semimodule over LA-semiring S and N be an LA-subsemimodule of M. Then, N cancellative and $0_M \in N$.*

Proof. The first statement is clear. The second statement, let a be an arbitrary element in M. Since N is an LA-subsemimodule of M and M is a cancellative LA-semimodule then $0_M = 0_S \cdot a \in N$. □

Theorem 2.2. *Let M be a cancellative LA-semimodule over LA-semiring S. If N_i are LA-subsemimodule of M for $i = 1, 2, 3, ..., n$, then $\bigcap_{i=1}^{n} N_i$ is an LA-subsemimodule of M.*

Proof. Since N_i is an LA-subsemimodule then $0_M \in N_i$ for all $i = 1, 2, 3, \ldots, n$. Hence, $\bigcap_{i=1}^{n} N_i \neq \emptyset$. Clear that $\bigcap_{i=1}^{n} N_i \subseteq M$. Let $a, b \in \bigcap_{i=1}^{n} N_i$, then $a, b \in N_i$ for all $i = 1, 2, 3, \ldots, n$. Since N_i are an LA-subsemimodule, then $a + b \in N_i$ and $sa \in N_i$ for all $s \in S$, $i = 1, 2, \ldots, n$. As a consequence $sa \in \bigcap_{i=1}^{n} N_i$. Hence $\bigcap_{i=1}^{n} N_i$ also an LA-subsemimodule of M. □

Theorem 2.3. *Let M be an LA-semimodule over LA-semiring S. If N_i are a subsemimodule of M for $i = 1, 2, \ldots, n$, then $\sum_{i=1}^{n} N_i$ is a subsemimodule.*

Proof. Since N_i is an LA-subsemimodule then $N_i \neq \emptyset$ for all $i = 1, 2, 3, \ldots, n$. As a consequence $\sum_{i=1}^{n} N_i \neq \emptyset$ and $\sum_{i=1}^{n} N_i \subseteq M$. Let $a, b \in \sum_{i=1}^{n} N_i$ where $a = a_1 + a_2 + ... + a_n$ and $b = b_1 + b_2 + ... + b_n$ with $a_i, b_i \in N_i$ for all $i = 1, 2, 3, \ldots, n$. Since N_i are LA-subsemimodule, then we have

$$\begin{aligned}
a + b &= (a_1 + a_2 + ... + a_n) + (b_1 + b_2 + ... + b_n) \\
&= ((a_1 + ... + a_{n-1}) + a_n) + ((b_1 + ... + b_{n-1}) + b_n) \\
&= ((a_1 + ... + a_{n-1}) + (b_1 + ... + b_{n-1})) + (a_n + b_n) \\
&= ((a_1 + ... + a_{n-2} + a_{n-1}) + (b_1 + ... + b_{n-2}) + b_{n-1}) + (a_n + b_n) \\
&= ((a_1 + ... + a_{n-2} + (b_1 + ... + b_{n-2})) + (a_{n-1} + b_{n-1}) + (a_n + b_n) \\
&= (((a_1 + b_1) + (a_2 + b_2)) + ...) + (a_n + b_n) \\
&= (a_1 + b_1) + (a_2 + b_2) + ... + (a_n + b_n).
\end{aligned}$$

Since $a_i + b_i \in N_i$ then $a + b \in \sum_{i=1}^{n} N_i$. Hence, $\sum_{i=1}^{n} N_i$ is an LA-subsemigroup. Next, let s be an arbitrary element in S, then

$$sa = s(a_1 + a_2 + ... + a_n) = sa_1 + sa_2 + \ldots + sa_n \in \sum_{i=1}^{n} N_i.$$

Hence $\sum_{i=1}^{n} N_i$ also LA-subsemimodule of M. □

Definition 2.4. *Let M be an LA-semimodule over LA-semiring S and N is an LA-subsemimodule of M. $M/N = \{a + N : a \in M\}$ is called a quotient LA-semimodule.*

Note that the binary operation in quotient LA-semimodule M/N are $'+'$ and $'\cdot'$. Since M is medial then we have

$$\begin{aligned}(a + N) + (b + N) &= (a + b) + (N + N)\\ &= (a + b) + N.\end{aligned}$$

Since M is an LA-semimodule over S, and N is an LA-subsemimodule then

$$s(a + N) = sa + sN = sa + N.$$

Since M contains left identity and M satisfy medial law then

$$N + (a + N) = (0 + N) + (a + N) = (0 + a) + (N + N) = a + N.$$

For any $a + N, b + N \in M/N$ and $s \in S$. Hence, N is left identity element in M/N.

Proposition 2.1. *Let M be an LA-semimodule over LA-semiring S and N be an LA-subsemimodule of M. If M is cancellative then M/N is cancellative.*

Proof. Let $a + N, b + N$ and $c + N$ be arbitrary elements in M/N then

$$\begin{aligned}(a + N) + (c + N) = (b + N) + (c + N) &\Rightarrow (N + N) + (c + a) = (N + N) + (c + b)\\ &\Rightarrow N + (c + a) = N + (c + b)\\ &\Rightarrow (0 + N) + (c + a) = (0 + N) + (c + b)\\ &\Rightarrow (0 + c) + (N + a) = (0 + c) + (N + b)\\ &\Rightarrow c + (N + a) = c + (N + b)\\ &\Rightarrow N + a = N + b\\ &\Rightarrow (0 + N) + a = (0 + N) + b\\ &\Rightarrow (a + N) + 0 = (b + N) + 0\\ &\Rightarrow (a + N) = (b + N).\end{aligned}$$

Thus, M/N is right cancellative. Since M/N is an LA-semigroup with left identity then M/N is left cancellative too. Therefore, M/N is cancellative. □

2.3 LA-Semimodule Homomorpishm

In this section, we give the definition of LA-semimodule homomorpishm and its basic properties.

Definition 2.5. *Let M, M' be LA-semimodule over LA-semiring S. A map $\varphi : M \to M'$ is called LA-semimodule homomorpishm if for any $s \in S$ and $m, n \in M$, satisfies the following conditions:*

i. $\varphi(m + n) = \varphi(m) + \varphi(n)$

ii. $\varphi(sm) = s\varphi(m)$

Corollary 2.2. *Let M, M' be LA-semimodule over LA-semiring S and map $\varphi : M \to M'$ be an LA-semimodule homomorphism. If M and M' are cancellative then $\varphi(0_M) = 0_{M'}$.*

Proof. Let a be an arbitrary element in M and $x \in M'$ where $x = \varphi(a)$, then

$$\begin{aligned}\varphi(a) = x &\Leftrightarrow 0_S\varphi(a) = 0_S \cdot x\\ &\Leftrightarrow \varphi(0_S \cdot a) = 0_{M'}\\ &\Leftrightarrow \varphi(0_M) = 0_{M'}.\end{aligned}$$

□

Remark 2.2 ([8])**.** *LA-semigroup M is an LA-group iff M is finite cancellative.*

Lemma 2.1. *Let M, M' be finite cancellative LA-semimodule over LA-semiring S and map $\varphi : M \to M'$ be an LA-semimodule homomorphism. If M and M' are finite cancellative then $\varphi(-a) = -\varphi(a)$ for all $a \in M$.*

Proof. Let a be an arbitrary element in M. Since M is finite cancellative then there exist $-a \in M$ such that $-a + a = 0_M$. Since φ is LA-semimodule homomorphism and M' is finite cancellative then

$$\begin{aligned}\varphi(-a + a) = \varphi(0_M) &\Leftrightarrow \varphi(-a) + \varphi(a) = 0_{M'}\\ &\Leftrightarrow \varphi(-a) + \varphi(a) - \varphi(a) = 0_{M'} - \varphi(a)\\ &\Leftrightarrow \varphi(-a) = -\varphi(a).\end{aligned}$$

□

Theorem 2.4. *Let M, M' be cancellative LA-semimodule over LA-semiring S and map $\varphi : M \to M'$ be an LA-semimodule homomorpishm, then the following conditions are holds:*

i. If P is an LA-subsemimodule of M, then $\varphi(P)$ is an LA-subsemimodule of M'.

ii. If Q is an LA-subsemimodule of M', then $\varphi^{-1}(Q)$ is an LA-subsemimodule of M.

Proof. Note that

$$\varphi(P) = \{x \in M' \mid x = \varphi(a),\ a \in P\}$$
$$\varphi^{-1}(Q) = \{a \in M \mid \varphi(a) \in Q\}.$$

Then, consider that

i. Since P is an LA-subsemimodule and φ is an LA-semimodule homomorphism then $P \neq \emptyset$ and $\varphi(P) \neq \emptyset$. Clear that $\varphi(P) \subseteq M'$. Let x, y be two arbitrary elements in $\varphi(P)$ where $x = \varphi(a), y = \varphi(b), a, b \in P$ then $x + y = \varphi(a) + \varphi(b) = \varphi(a + b)$. Since P is an LA-subsemimodule then $a + b \in P$. Hence, $x + y \in \varphi(P)$ and $\varphi(P)$ is an LA-subsemigroup. Let s be an arbitrary element in S, then $sx = s\varphi(a) = \varphi(sa)$. Since P is an LA-subsemimodule then $sa \in P$. So, $sx \in \varphi(P)$. Therefore, $\varphi(P)$ is an LA-subsemimodule of M'.

ii. Since Q is an LA-subsemimodule of M' and M' is a cancellative LA-semimodule then $0_{M'} \in Q$. Hence, $0_{M'} = \varphi(a)$ implies $a = 0_M$, then $\varphi^{-1}(Q) \neq \emptyset$. Clear that $\varphi^{-1}(Q) \subseteq M$. Let a, b be two arbitrary element in $\varphi^{-1}(Q)$, then $\varphi(a + b) = \varphi(a) + \varphi(b) \in Q$. Hence, $a + b \in \varphi^{-1}(Q)$ and $\varphi^{-1}(Q)$ is an LA-subsemigroup. Let $s \in S$ then $\varphi(sa) = s\varphi(a)$. Since $\varphi(a) \in Q$ and Q is an LA-subsemimodule then $s\varphi(a) \in Q$. Therefore, $\varphi^{-1}(Q)$ is an LA-subsemimodule of M.

□

Definition 2.6. *Let $\varphi : M \to M'$ be an LA-semimodule homomorpishm. Kernel of φ is defined by $Ker(\varphi) = \{m \in M : \varphi(m) = 0\}$ and image of φ is defined by $Im(\varphi) = \{\varphi(m) : m \in M\}$.*

Lemma 2.2. *Let M, M' be cancellative LA-semimodule over LA-semiring S and map $\varphi : M \to M'$ be an LA-semimodule homomorpishm, then $Ker(\varphi)$ and $Im(\varphi)$ are LA-subsemimodule of M and M', respectively.*

Proof. Since $\varphi(0_M) = 0_{M'}$ then $Ker(\varphi) \neq \emptyset$. Clear that $Ker(\varphi) \subseteq M$. Let a, b be two arbitary element in $Ker(\varphi)$ then $\varphi(a + b) = \varphi(a) + \varphi(b) = 0_{M'}$. Hence, $a + b \in Ker(\varphi)$ and $Ker(\varphi)$ is an LA-subsemigroup of M. Let s be an arbitary element in S, then $\varphi(sa) = s\varphi(a) = s \cdot 0_{M'} = 0_{M'}$. Therefore, $sa \in Ker(\varphi)$ and $Ker(\varphi)$ is an LA-subsemimodule of M.

Next, since $\varphi(0_M) = 0_{M'}$, then $Im(\varphi) \neq \emptyset$. Clear that $Im(\varphi) \subseteq M'$. Let x, y be two arbitary element in $Im(\varphi)$ where $x = \varphi(a), y = \varphi(b)$, $a, b \in M$ then $x + y = \varphi(a) + \varphi(b) = \varphi(a + b)$. Since $a + b \in M$ then $x + y \in Im(\varphi)$ and $Im(\varphi)$ is an LA-subsemigroup of M'. Let s be an arbitrary element in S then $sx = s\varphi(a) = \varphi(sa)$. Since M is an LA-subsemimodule then $sa \in M$. Hence, $sx \in Im(\Phi)$ and $Im(\Phi)$ is an LA-subsemimodule of M'. □

Proposition 2.2. *Let M, M' be finite cancellative LA-semimodule over LA-semiring S and map $\varphi : M \to M'$ be an LA-semimodule homomorpishm. Map φ is one-one if and only if $Ker(\varphi) = \{0_M\}$.*

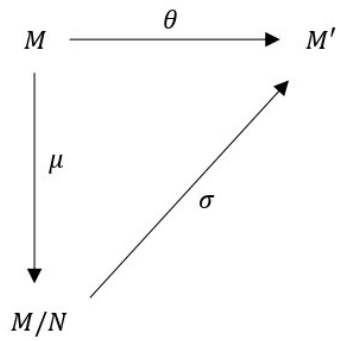
Proof. ($\Rightarrow$) Let a be an arbitrary element in $Ker(\varphi)$, then $\varphi(a) = 0_{M'}$. Since M, M' are cancellative LA-semimodule and φ is an LA-semimodule homomorphism then $\varphi(0_M) = 0_{M'}$. Since φ is one-one and $\varphi(a) = 0_{M'} = \varphi(0_M)$, then $a = 0_M$. Finally, $ker(\varphi) = \{0_M\}$.

($\Leftarrow$) Let $Ker(\varphi) = \{0_M\}$ and a, b be two arbitrary elements in M such that $\varphi(a) = \varphi(b)$. Since M, M' are finite cancellative LA-semigroup, then M, M' are LA-group. Hence, $\varphi(a) = \varphi(b)$ implies $\varphi(a) - \varphi(b) = 0_{M'}$. Since φ is an LA-semimodule homomorpishm, then $\varphi(a - b) = 0_{M'}$. Hence, $a - b \in Ker(\varphi)$. Since $Ker(\varphi) = \{0_M\}$ then $a - b = 0_M$. As consequence, $a = b$. So, φ is one-one. □

2.4 Isomorpishm Theorem for LA-Semimodule

In this section, we discuss about the isomorphism theorem in LA-semimodule over LA-semiring.

Theorem 2.5. *Let M, M' be finite cancellative LA-semimodule over LA-semiring S, $\theta : M \to M'$ be an LA-semimodule epimorphism, and $\mu : M \to M/Ker(\theta)$ be a natural LA-semimodule homomorphism. Then, there exist an LA-semimodule isomorphism $\sigma : M/N \to M'$, where $N = Ker(\theta)$ and make the diagram below commute.*



Proof. Since M, M' is a cancellative LA-semimodule and θ is an LA-semimodule homomorphism then $N = Ker(\theta)$ is an LA-subsemimodule of M. Therefore, M/N is a quotient LA-semimodule. Next, consider the mapping

$$\begin{aligned} \sigma : M/N &\to M' \\ a + N &\mapsto \sigma(a + N) = \theta(a) = a'. \end{aligned}$$

Then, we will show that σ is an isomorphism. Note that,

i. First, we will show that the mapping is well defined. Since M and M' are finite cancellative LA-semimodule then M/N is a finite cancellative quotient LA-semimodule. Hence, M/N and M' are LA-semigroup. Let $a + N, b + N$ be two arbitrary elements in M/N such that $a + N = b + N$, then

$$\begin{aligned} a + N = b + N &\Rightarrow a - b + N = N \\ &\Rightarrow a - b \in N \\ &\Rightarrow \theta(a - b) = 0 \\ &\Rightarrow \theta(a) - \theta(b) = 0 \\ &\Rightarrow \theta(a) = \theta(b) \\ &\Rightarrow \sigma(a + N) = \sigma(b + N). \end{aligned}$$

Thus σ is well defined.

ii. Let $a + N$ and $b + N$ be two arbitrary elements of M/N such that $\sigma(a + N) = \sigma(b + N)$, then

$$\begin{aligned} \sigma(a + N) = \sigma(b + N) &\Rightarrow \theta(a) = \theta(b) \\ &\Rightarrow \theta(a) - \theta(b) = 0 \\ &\Rightarrow \theta(a - b) = 0 \\ &\Rightarrow a - b \in N \\ &\Rightarrow a - b + N = N \\ &\Rightarrow a + N = b + N. \end{aligned}$$

Therefore, σ is one-one.

iii. Next we will show that σ is onto. Let a' be an arbitary element of M'. Since θ is an epimorphism from M to M', then there is an element a in M such that $\theta(a) = a'$. Since $\theta(a)$ being the σ-image of the coset $a + N$ in M/N, then $a' = \theta(a) = \sigma(a + N)$. Thus, σ is onto.

iv. Finally, σ is an LA-semimodule homomorpishm, i.e

(a) Let $a + N, b + N$ be two arbitrary elements in M/N then

$$\begin{aligned}\sigma[(a+N)+(b+N)] &= \sigma[(a+b)+N] \\ &= \theta(a+b) \\ &= \theta(a)+\theta(b) \\ &= \sigma(a+N)+\sigma(b+N).\end{aligned}$$

(b) Let $a + N$ be an arbitrary element in M/N and s be an arbitrary element in S then

$$\begin{aligned}\sigma[s(a+N)] &= \sigma(sa+N) \\ &= \theta(sa) \\ &= s\theta(a) \\ &= s\sigma(a+N).\end{aligned}$$

Hence, σ is an LA-semimodule isomorpishm from M/N to M' or $M/N \cong M'$. □

Theorem 2.6. *Let M be a finite cancellative LA-semimodule over LA-semiring S. If I and J are LA-subsemimodule of M, then $\frac{I+J}{J} \cong \frac{I}{I\cap J}$.*

Proof. Since I and J are LA-subsemimodule of M, then $I+J$ is an LA-subsemimodule of M. Since $J \subseteq I + J$ and J is an LA-subsemimodule, then $\frac{I+J}{J}$ is a quotient LA-semimodule. Since I and J are LA-subsemimodule, then $I \cap J$ is an LA-subsemimodule. Since $I \cap J \subseteq I$, then $\frac{I}{I\cap J}$ is a quotient LA-semimodule. Next, define a mapping

$$\begin{aligned}\theta : I &\to \frac{I+J}{J} \\ a &\mapsto \theta(a) = (a+0)+J = a+J.\end{aligned}$$

We will prove this theorem by using Theorem 2.5, then note that

i. Clear that φ is well defined. Let a, b be two arbitrary elements in I and s be an arbitrary element in S, then

(a) $\theta(a + b) = (a + b) + J = (a + J) + (b + J) = \theta(a) + \theta(b)$.
(b) $\theta(sa) = (sa) + J = s(a + J) = s\theta(a)$.

Thus, θ is an LA-semimodule homomorpishm. Next, note that for any $a + J \in \frac{I+J}{J}$, then exists $a \in I$ such that $\theta(a) = a + J$. Therefore, θ is an onto homomorpishm.

ii. Since J is a left identity in quotient LA-semimodule $\frac{I+J}{J}$, then

$$\begin{aligned}Ker(\theta) &= \{a \in I : \theta(a) = J\} \\ &= \{a \in I : a+J = J\} \\ &= \{a \in I : a \in J\} \\ &= \{a \in I \cap J\} \\ &= I \cap J.\end{aligned}$$

Since θ is an LA-semimodule epimorphism, $Ker(\theta) = I \cap J$ and M is finite cancellative then by Theorem 2.5 we have $\frac{I+J}{J} \cong \frac{I}{I \cap J}$. □

Theorem 2.7. *Let M be a finite cancellative LA-semimodule over LA-semiring S. If J and K are LA-subsemimodule of M, where $J \subseteq K$, then $\frac{M/J}{K/J} \cong \frac{M}{K}$.*

Proof. Clear that M/J, M/K and K/J are quotient LA-semimodule over S. Since $K \subseteq M$ then $K/J \subseteq M/J$. Hence, K/J is an LA-subsemimodule of M/J and $\frac{M/J}{K/J}$ is an quotient LA-semimodule. Define a mapping

$$\begin{aligned} \theta : M/J &\to M/K \\ a + J &\mapsto \theta(a+J) = a + K. \end{aligned}$$

Then, note that

i. Since M is cancellative then M/J and M/K are cancellative. Hence, $\theta(J) = K$ implies θ is well defined. Then, we will show that θ is an LA-semimodule homomorphism

(a) For any $a + J, b + J \in M/J$, we have

$$\begin{aligned} \theta[(a+J)+(b+J)] &= \theta[(a+b)+J] \\ &= (a+b)+K \\ &= (a+K)+(b+K) \\ &= \theta(a+J)+\theta(b+J). \end{aligned}$$

(b) For any $a + J \in M/J$ and $s \in S$, we have

$$\begin{aligned} \theta[s(a+J)] &= \theta(sa+J) \\ &= sa+K \\ &= s(a+K) \\ &= s\theta(a+J). \end{aligned}$$

Hence, θ is an LA-semimodule homomorpishm.

Furthermore, since $J \subseteq K$ then for any $a + K \in M/K$, we can choose $a + J \in M/J$ such that $\theta(a + J) = a + K$. Thus, θ is an epimorphism.

ii. We will show that $Ker(\theta) = K/J$, then consider that

$$\begin{aligned} Ker(\theta) &= \{a+J \in M/J : \theta(a+J) = K\} \\ &= \{a+J \in M/J : a+K = K\} \\ &= \{a+J \in M/J : a \in K\} \\ &= \{a+J \in K/J\} = K/J. \end{aligned}$$

Now, since θ is an LA-semimodule epimorphism, $Ker(\theta) = K/J$ and $M/J, M/K$ are finite cancellative, then by Theorem 2.5 we have $\frac{M/J}{K/J} \cong \frac{M}{K}$. □

3 Conclusions

Any LA-semimodule over LA-semiring are satisfy The First Isomorphism Theorem, The Second Isomorphism Theorem and The Third Isomorphism Theorem. Also, LA-semimodule over LA-semiring are satisfy some properties like properties of module over ring.

Actknowledgement The authors gratefully acknowledge various helpful comments and suggestions made by the reviewers.

Conflict of Interest The authors declare no conflict of interest.

References

[1] J. R. Cho, J. Jezek & T. Kepka (1999). Paramedial groupoids. *Czechoslovak Mathematical Journal*, *49*(2), 277–290.

[2] J. S. Golan (2013). *Semirings and their applications*. Springer Science & Business Media, Haifa, Israel.

[3] M. S. Kamran (1993). *Conditions for LA-Semigroups To Resemble Associative Structures*. PhD thesis, Quad-i-Azam University, Islamabad, Pakistan.

[4] M. A. Kazim & M. Naseerudin (1977). On almost semigroups. *Portugaliae Mathematica*, *36*, 41–47.

[5] Q. Mushtaq & M. S. Kamran (1996). On left almost groups. *Proceedings of the Pakistan Academy of Sciences*, *33*, 1–2.

[6] Q. Mushtaq & S. M. Yusuf (1979). On locally associative LA-semigroups. *Journal of Natural Sciences and Mathematics*, *XIX*(1), 57–62.

[7] K. Rahman, F. Husain, S. Abdullah & M. S. A. Khan (2016). Left almost semirings. *International Journal of Computer Science and Information Security*, *14*(9), 201–216.

[8] M. Shah, T. Shah & A. Ali (2011). On the cancellativity of AG-groupoids. *International Mathematical Forum*, *6*(44), 2187–2194.

[9] M. Shah & T. Shah (2011). Some basic properties of LA-ring. *International Mathematical Forum*, *6*(44), 2195–2199.

[10] T. Shah, M. Raees & G. Ali (2011). On LA-modules. *International Journal of Contemporary Mathematical Sciencess*, *6*(21), 999–1006.